CDS 131 Homework 1: Linear Dynamical Systems

Winter 2025

Due 1/13 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Systems of First Order Equations

1. Show that an n'th order linear ODE,
$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^n} + ... + a_1\frac{dx}{dt} + a_0 x = 0,\ a_i \in \mathbb{R}, \tag{1}$$
can be rewritten as a system of n, first order differential equations of the form,
$$\dot{z} = Az, \tag{2}$$
where $z \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. This tells us that it's sufficient to examine *linear systems of first order ODEs* in order to reach conclusions about linear n'th order ODEs.

2. Show that an n'th order recurrence,
$$x[k+n] + a_{n-1}x[k+n-1] + ... + a_1 x[k+1] + a_0 x[k] = 0, \tag{3}$$
can be rewritten as a system of n, first order recurrences of the form,
$$z[k+1] = Az[k], \tag{4}$$
where $z \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. This tells us that it's sufficient to examine *linear systems of first order recurrences* in order to reach conclusions about linear n'th order recurrences.

1.2 Practice with Linear ODEs

1. Determine the state transition matrix for the linear system $\dot{x}(t) = A(t)x(t)$, where
$$A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}, \tag{5}$$
by either (a) directly solving differential equations or (b) using the Peano-Baker series.

1.3 Practice with Linear Recurrences

1. Determine the state transition matrix for the discrete-time recurrence, $x[k+1] = A[k]x[k]$, where

$$A[k] = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

Hints: (1) How does the state transition matrix simplify in the case where A is constant? (2) Use the eigendecomposition of A to more easily compute A^k.

2 Required Problems

2.1 Properties of Piecewise Continuous Functions

This week, we introduced the class of piecewise continuous functions as a class of signals for continuous-time systems. In this problem, we'll prove some basic properties of this function class.

1. Let $I \subseteq \mathbb{R}$ be a compact interval. Show that $PC(I, \mathbb{R}^n)$ forms a vector space over $\mathbb{R}$ under the operations of function addition and scalar multiplication. *Hint: prove it is a subspace of another function space to make your life a little easier!*

2. Let $I, K \subseteq \mathbb{R}$ be compact intervals. Show that any $f \in PC(I, \mathbb{R})$ must be bounded above on $I \cap K$,

$$\sup_{t \in I \cap K} f(t) < \infty. \tag{7}$$

3. Let $I \subseteq \mathbb{R}$ be a compact interval and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Show that the supremum norm,

$$\|f\|_\infty = \sup_{t \in I} \|f(t)\|, \tag{8}$$

is finite for all $f \in PC(I, \mathbb{R}^n)$. Then, prove that $\|\cdot\|_\infty$ makes $PC(I, \mathbb{R}^n)$ into a normed vector space.

4. Is $PC(I, \mathbb{R}^n)$ a Banach space with respect to the supremum norm $\|\cdot\|_\infty$, $\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f(t)\|$? Provide a proof or a counterexample.

2.2 Transition Matrix Under Change of Variables

Consider a continuous-time linear, time-varying system representation $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{9}$$

$$y(t) = C(t)x(t) + D(t)u(t). \tag{10}$$

1. Consider an invertible linear transformation $T \in \mathbb{R}^{n \times n}$ and a corresponding change of variables, $z = Tx$. Identify the system representation $(\hat{A}(\cdot), \hat{B}(\cdot), \hat{C}(\cdot), \hat{D}(\cdot))$ for which solutions to,

$$\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)u(t) \tag{11}$$

$$\hat{y}(t) = \hat{C}(t)z(t) + \hat{D}(t)u(t) \tag{12}$$

satisfy $z(t) = Tx(t)$ and $\hat{y}(t) = y(t)$ for all initial conditions x_0 and Tx_0 and piecewise continuous input signals $u(\cdot)$. Conclude that the input to output behavior of the system *does not* depend on changes of state coordinates.

2. Write the state transition matrix $\hat{\Phi}(t, t_0)$ of the transformed system in terms of the state transition matrix $\Phi(t, t_0)$ of the original system and the transformation T.

3. Does the relation you derived in part (2) also hold for a discrete-time system representation? Explain why or why not.

2.3 An Inverse Initial Value Problem

We know that $\Phi(t, t_0)$ is the solution to the initial value problem $\dot{X}(t) = A(t)X(t),\ X(t_0) = I$. In this problem, we'll find out what $\Phi(t_0, t)$ corresponds to.

1. Consider a continuously differentiable, matrix-valued function $M(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$. Suppose for all $t \in \mathbb{R}$, $M(t)$ is nonsingular. Determine an expression for $\frac{d}{dt}[M^{-1}(t)]$ in terms of $\dot{M}(t)$ and $M^{-1}(t)$.

2. Now, consider a matrix $A(\cdot) \in PC(\mathbb{R}, \mathbb{R}^{n\times n})$. Find an expression for the derivative $\frac{\partial}{\partial\tau}\Phi(t, \tau)$ of the state transition matrix Φ with respect to $A(\cdot)$, in terms of $\Phi(t, t_0)$ and $A(t)$. *You may assume the derivative is being taken at a point where $A(\cdot)$ is continuous.*

3. Prove that $\Phi(t_0, t)$ is the unique solution of the matrix initial value problem,

$$\dot{X}(t) = -X(t)A(t),\ X(t_0) = I. \tag{13}$$

2.4 The Jacobi-Liouville Formula

This week, we showed that the state transition matrix is always invertible. Here, we'll provide another proof of this by means of the *Jacobi-Liouville formula*, which explicitly provides a formula for the determinant of the state transition matrix. In particular, the Jacobi-Liouville formula is,

$$\det \Phi(t, t_0) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(\tau))d\tau\right). \tag{14}$$

1. Prove that, for $M \in \mathbb{R}^{n\times n}$ and $\epsilon \in \mathbb{R}$, there exists a continuous function $R : \mathbb{R} \to \mathbb{R}$ for which

$$\det(I + \epsilon M) = 1 + \epsilon \operatorname{tr}(M) + R(\epsilon) \text{ and } \lim_{\epsilon\to 0} \frac{R(\epsilon)}{\epsilon} = 0. \tag{15}$$

Hint: consider working with eigenvalues.

2. Using the determinant formula from (1), show that

$$\frac{d}{dt}\det[\Phi(t, t_0)] = \operatorname{tr}(A(t))\det[\Phi(t, t_0)]. \tag{16}$$

Hint: Work with the limit definition of the derivative. If you use a Taylor approximation, be rigorous about your use of the remainder term.

3. Conclude the Jacobi-Liouville formula. Using the Jacobi-Liouville formula, provide a proof that $\Phi(t, t_0)$ is invertible for all $(t, t_0) \in \mathbb{R} \times \mathbb{R}$.

2.5 A Special State Transition Matrix

Consider a piecewise continuous matrix $A \in PC(\mathbb{R}, \mathbb{R}^{n\times n})$, and let Φ denote the state transition matrix of $\dot{x}(t) = A(t)x(t)$. If for every $(\tau, t) \in \mathbb{R} \times \mathbb{R}$, one has,

$$A(t)\left(\int_{\tau}^{t} A(\eta)d\eta\right) = \left(\int_{\tau}^{t} A(\eta)d\eta\right)A(t), \tag{17}$$

prove using the Peano-Baker series that,

$$\Phi(t, \tau) = \exp\left(\int_{\tau}^{t} A(\eta)d\eta\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{\tau}^{t} A(\eta)d\eta\right)^{k}. \tag{18}$$

Using this result, calculate the state transition matrix associated to the matrix,

$$A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}. \tag{19}$$

3 Optional Problems

3.1 Causal & Noncausal Maps

One can represent the input/output relationship of a system for a *fixed* initial time and initial state with a function $H : \mathcal{T} \times \mathcal{U} \to Y$. That is, one has $y(t) = H(t, u(\cdot))$ for any time t and admissible input $u(\cdot)$. In this problem, we'll determine definitions for causality, linearity, and time-invariance of an arbitrary map $H : \mathcal{T} \times \mathcal{U} \to Y$ between a time set $\mathcal{T}$, a set of input signals $\mathcal{U}$, and a set of output values Y.

1. Given an arbitrary map $H : \mathcal{T} \times \mathcal{U} \to Y$, formulate a definition of *time-invariance* for H. Formulate a definition of *causality.* Formulate a definition of *linearity. Hint: for causality, think about the restriction of a signal to a certain time interval.*

2. Let's put our definitions to the test. In each of the following cases, determine whether the system is causal/time-invariant/linear. Use your best judgment to identify the time set and input and output spaces in each case.

    (a) Consider a discrete-time system with I/O description $y[k] = c_1 u[k+1] + c_2$, where $c_1, c_2 \in \mathbb{R}$. Is this system causal? Is it time-invariant? Is it linear?

    (b) Consider a continuous-time system with I/O description $y(t) = u(t - \tau)$, where $\tau \in \mathbb{R}$ is fixed and positive. Is this system causal? Is it time invariant? Is it linear?

    (c) Consider a continuous-time system with I/O description,

    $$y(t) = \begin{cases} u(t) & t \leq \tau \\ 0 & t > \tau, \end{cases} \tag{20}$$

    where $\tau \in \mathbb{R}$ is fixed. Is this system causal? Is it time-invariant? Is it linear?

    (d) Consider a continuous-time system with I/O description,

    $$y(t) = \min\{u_1(t), u_2(t)\}, \tag{21}$$

    where $u(t) = [u_1(t); u_2(t)]^\top$ is the system input. Is this system causal? Is it time-invariant? Is it linear?

3.2 Solution of a Matrix Differential Equation

Let $A_1(\cdot), A_2(\cdot)$, and $F(\cdot)$ be elements of $PC(\mathbb{R}, \mathbb{R}^{n \times n})$. Let Φ_i be the state transition matrix of $\dot{x}(t) = A_i(t)x(t)$ for $i = 1, 2$. Show that the solution of the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^\top(t) + F(t),\ X(t_0) = X_0, \tag{22}$$

is given by,

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^\top(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)F(\tau)\Phi_2^\top(t, \tau)d\tau. \tag{23}$$

Is this the unique solution of the matrix differential equation? Back up your answer with a proof or disproof.