CDS 131 Homework 2: LTI and I/O Systems

Winter 2025

Due 1/21 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Fun with Jordan Forms

In the following problem, we'll get some practice with Jordan forms.

1. Find a Jordan canonical form of each of the following matrices,

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}. \tag{1}$$

2. Calculate the matrix exponentials $\exp(A_i t),\ i = 1, 2$.
3. For each A_i, illustrate how an arbitrarily small numerical perturbation can change the structure of the Jordan form. How do you think this affects numerical computation of the Jordan form?

1.2 Algebraic Properties of the Matrix Exponential

Let $A, B \in \mathbb{R}^{n\times n}$. Let $A \in \mathbb{R}^{n\times n}$. Prove the following properties of the matrix exponential:

1. For every $t_1, t_0 \in \mathbb{R}$, $\exp(A(t_1 + t_0)) = \exp(At_1)\exp(At_0) = \exp(At_0)\exp(At_1)$.
2. For an eigenvalue-vector pair (λ, v) of A, (e^λ, v) is an eigenvalue-vector pair of $\exp(A)$.
3. $\det(\exp A) = e^{\operatorname{tr} A}$.
4. $(\exp(A))^{-1} = \exp(-A)$.

Hint: read the result of the required Problem 2.2 (below) regarding commutativity & the exponential before attempting these problems.

1.3 Some Laplace & $\mathcal{Z}$-Transforms

In this problem, we'll establish a couple of basic Laplace and $\mathcal{Z}$-transforms.

1. Let f be a signal and $\tau > 0$. Define the signal g,

$$g(t) = \begin{cases} 0, & 0 \leq t < \tau. \\ f(t-\tau), & t \geq \tau. \end{cases} \tag{2}$$

   Show that $\hat{G}(s) = e^{-s\tau}\hat{F}(s)$.

2. Show that the $\mathcal{Z}$-transform of the unit step function,

$$\mathbb{1}[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0, \end{cases} \tag{3}$$

   is $\hat{\mathbb{1}}(z) = z/(z-1)$.

2 Required Problems

2.1 Commutativity & The Exponential

Let $A, B \in \mathbb{R}^{n \times n}$. In this problem, we'll prove that commutativity of A and B, $AB = BA$, implies $\exp(A+B) = \exp(A)\exp(B)$. Notably, we'll give a proof that uses an *existence and uniqueness* argument, rather than a direct algebraic argument.

1. Give examples of matrices A, B for which $\exp(A+B) \neq \exp(A)\exp(B)$. *Feel free to use a computational tool to experiment with different matrices.*

2. Using an existence and uniqueness argument, prove that $AB = BA$ implies $\exp(A+B) = \exp(A)\exp(B)$. *Hint: set up an initial value problem involving A and B.*

2.2 Schur Triangulation

The Jordan normal form—though convenient for theoretical calculations—suffers from a number of numerical problems. Here, we consider an alternative technique for computing the matrix exponential, based on the *Schur triangulation* of a matrix.

1. A matrix $T \in \mathbb{C}^{n \times n}$ is said to be *unitary* if $TT^* = T^*T = I$. Prove the *Schur triangulation theorem*, which states that, for any $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix T for which $U := T^*AT$ is upper triangular. *Hints: Proceed by induction.*

2. Determine a method for computing the matrix exponential of an upper triangular matrix. *Your method does not have to be computationally efficient, it just needs to work.*

3. Comment on the benefits and drawbacks of using your Schur triangulation method versus the Jordan normal form method of calculating the matrix exponential.

If you're interested in reading about more ways of computing the matrix exponential, check out the paper *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*.

2.3 The Floquet Decomposition

Consider the system $\dot{x}(t) = A(t)x(t)$, in which $A(\cdot)$ is periodic with period $T > 0$, $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. The basic idea of the *Floquet decomposition* is that, by constructing a time-varying transformation that "syncs up" with the periodicity of $A(\cdot)$, we can use time-invariant tools to study a time-varying system.

1. Let $\Phi(t, t_0)$ denote the state transition matrix of $\dot{x}(t) = A(t)x(t)$. Show that $\Phi(t+T, 0) = \Phi(t, 0)\Phi(T, 0)$.

2. Prove that for every nonsingular matrix $B \in \mathbb{C}^{n\times n}$, there exists a matrix $A \in \mathbb{C}^{n\times n}$ for which $\exp(A) = B$. *Hint: the complex (scalar) logarithm is defined on the nonzero complex numbers.*

3. Prove there exists an $R \in \mathbb{C}^{n\times n}$ for which $\Phi(T, 0) = \exp(TR)$.

4. Consider a time-varying transformation $P : \mathbb{R} \to \mathbb{C}^{n\times n}$, for which

$$P(t)^{-1} = \Phi(t, 0)e^{-tR}. \tag{4}$$

Show that $P(t)$ is in fact invertible for all t. Then, prove that for all $t, t_0 \in \mathbb{R}$,

$$\Phi(t, t_0) = P(t)^{-1}e^{R(t-t_0)}P(t_0). \tag{5}$$

Comment on the significance of this result. *The eigenvalues of the R matrix, called Floquet multipliers, help in determining the stability of periodic systems.*

2.4 Zero-Order Hold Discretization

In this problem, we'll show how a continuous-time linear can be *exactly* discretized into a discrete-time linear system. Consider the continuous-time system representation $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{6}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{7}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. Consider a strictly increasing sequence of sampling times, $\{t_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, satisfying $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$. Suppose the input signals $u(\cdot)$ to the continuous-time system are constant on the sampling intervals. That is, for every input signal $u(\cdot)$ to the continuous-time system, there exists a sequence $u[\cdot] : \mathbb{Z} \to \mathbb{R}^m$ for which $u(t) = u[k] \in \mathbb{R}$ for all $t \in [t_k, t_{k+1})$.

1. Show there exists a discrete-time system representation $(\hat{A}[\cdot], \hat{B}[\cdot], \hat{C}[\cdot], \hat{D}[\cdot])$ such that for all initial conditions $x_0 = x(t_0) = x[0] \in \mathbb{R}^n$, solutions to the system,

$$x[k + 1] = \hat{A}[k]x[k] + \hat{B}[k]u[k] \tag{8}$$

$$y[k] = \hat{C}[k]x[k] + \hat{D}[k]u[k], \tag{9}$$

satisfy $x[k] = x(t_k)$ and $y[k] = y(t_k)$ for all $k \in \mathbb{Z}$, where $x(t_k)$ and $y(t_k)$ are the state and output of the continuous-time system at time t_k. This tells us that we can *exactly* discretize the continuous-time LTV system.

2. Now, suppose each matrix in the continuous-time system representation is constant,

$$(A(\cdot), B(\cdot), C(\cdot), D(\cdot)) = (A, B, C, D). \tag{10}$$

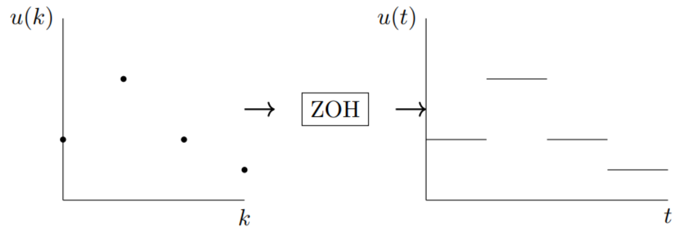
Further, assume that for each $k \in \mathbb{Z}$, $t_{k+1} - t_k = \Delta$. Using your answer to (1), show that there exists a discrete-time LTI system representation $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ which exactly discretizes the continuous-time LTI system.

Discretization in which one holds an input signal constant across a sampling period is referred to as *zero-order hold (ZOH)* discretization.

2.5 Sampled-Data Systems

In this problem, we'll take a frequency-domain approach to zero-order hold discretization. Consider a continuous-time system where inputs and outputs can only be accessed at discrete times $t = k\Delta$, $k \in \mathbb{Z}$, with a sampling period $\Delta \in \mathbb{R}_{>0}$. The discrete-time input $u[k] = u(k\Delta)$ is passed through a zero-order hold (ZOH) digital-to-analog (D/A) converter that accepts the input $u(k\Delta)$ at $t = k\Delta$ and holds it constant until the next input is applied at $t = (k + 1)\Delta$. The continuous-time system processes the ZOH output, and its resulting continuous-time output is sampled by the A/D converter to produce the discrete-time output $y[k]$. The goal is to compute the discrete-time transfer function of the overall system considering the effects of D/A and A/D converters.

1. First, we will compute the transfer function of the zero order hold. The zero-order hold takes in a continuous-time input signal (which has been sampled with interval Δ) and returns a continuous-time *held* version of the signal.

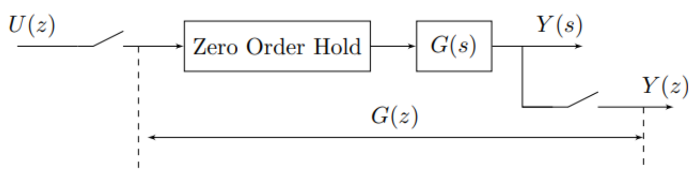


Show that the transfer function of the zero-order hold is

$$H(s) = \frac{1 - e^{-s\Delta}}{s}. \tag{11}$$

Assume that the sampler translates a discrete-time unit impulse to a continuous-time Dirac delta.

2. Now, we apply the zero-order hold block to a system. Consider the block diagram,



where $G(s)$ is the transfer function of a continuous-time system and the latches represent sampling. Show that the transfer function from $U(z)$ to $Y(z)$ is,

$$G(z) = (1 - z^{-1})\mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right]. \tag{12}$$

Hint: Split $H(s)G(s)$ into two components. How do the two components relate?

3. If the continuous-time transfer function is,

$$G(s) = \frac{a}{s^2}, \tag{13}$$

what is the corresponding discrete-time transfer function $G(z)$?

3 Optional Problems

Fundamentally, the construction of the Jordan canonical form relies on the fact that the generalized eigenspaces are *invariant subspaces* - subspaces $V \subseteq \mathbb{C}^n$ satisfying $AV \subseteq V$. In this problem, we'll study some basic properties of invariant subspaces and see how they relate to the Jordan form.

1. Let $T : V \to V$ be a linear transformation on an n-dimensional vector space V over $\mathbb{K}$. A subspace $M \subseteq V$ is said to be T-invariant if $Tx \in M$ for all $x \in M$. Suppose that V is the direct sum of two subspaces $M_1, M_2 \subseteq V$, $V = M_1 \oplus M_2$. If both M_1 and M_2 are T-invariant, prove there exists a basis β for V in which T has the matrix representation,

$$[T]_\beta = A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix} \in \mathbb{K}^{n \times n}, \tag{14}$$

where $\dim(M_1)$ and $\dim(A_2)$ equal the sizes of A_{11} and A_{12}. Argue that the restrictions $T|_{M_1} : M_1 \to M_1$ and $T|_{M_2} : M_2 \to M_2$ are well-defined maps.

2. We define a generalized eigenspace of the linear transformation T to be a space,

$$K_\lambda(T) = \{v \in V : (T - \lambda I)^m = 0 \text{ for some } m \in \mathbb{N}\} \subseteq V, \tag{15}$$

where λ is an eigenvalue of T. Prove the following:

(a) $K_\lambda(T)$ contains at least one eigenvector of T.

(b) $K_\lambda(T)$ is a subspace.

(c) $K_\lambda(T)$ is T-invariant.

(d) $K_\lambda(T) = \{v \in V : (T - \lambda I)^{\dim V} = 0\}$.

3. Suppose T has two distinct eigenvalues, λ_1 and λ_2, and $V = K_{\lambda_1}(T) \oplus K_{\lambda_2}(T)$. Suppose for each $i = 1, 2$, the sets

$$\beta_i = \{v_i, (T - \lambda_i I)v_i, ..., (T - \lambda_i I)^{m_i - 1} v_i\}, \tag{16}$$

form bases for K_{λ_1} and K_{λ_2}. Construct a basis β for V in which T is in Jordan canonical form.