CDS 131 Homework 3: Transforms & Stability in State Space

Winter 2025

Due 1/27 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Transforms & Transition Matrices

The Laplace transform offers yet another way of computing the state transition matrix. For a continuous-time, LTI representation (A, B, C, D), the matrix exponential is computed $\exp(At) = \mathcal{L}^{-1}[(sI - A)^{-1})](t)$, for all $t \geq 0$. In this problem, we'll consider an analogue in discrete-time, and use both the continuous and discrete formulas to compute some transition matrices.

1. Show that the state transition matrix of a discrete-time, LTI representation (A, B, C, D) is computed,

$$A^k = \mathcal{Z}^{-1}[z(zI - A)^{-1}][k], \ \forall k \geq 0. \tag{1}$$

2. Using the transform formulas, compute the continuous and discrete-time transition matrices associated to the matrix,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}. \tag{2}$$

   Comment on the benefits and drawbacks of this method of computing the transition matrix. You may use a symbolic calculator to compute the inverse of $(sI - A)$.

1.2 Transfer Functions & Change of Basis

Consider a linear, time-invariant system representation (A, B, C, D). Recall that under a change of state coordinates, $z = Tx$, the representation *transforms* to $(TAT^{-1}, TB, CT^{-1}, D)$. Does the transfer function associated to the system representation change under a change of state coordinates? Provide a proof or counterexample to back up your answer.

1.3 Analytic Functions

Recall that a given function $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is open in $\mathbb{C}$, is an *analytic function* if it is (complex) differentiable in a neighborhood of every point of $\mathbb{C}$. For each of the *scalar* functions of $s \in \mathbb{C}$,

$$f_1(s) = \frac{1}{s},\ f_2(s) = e^s,\ f_3(s) = \frac{(s-1)}{(s+1)(s-1)(s+2)},\ G(s) = C(sI - A)^{-1}B, \tag{3}$$

determine the largest subset of $\mathbb{C}$ on which the function is analytic.

2 Required Problems

2.1 A Simple SISO Transfer Function

1. Consider a continuous-time SISO, LTI system representation (A, B, C, D),

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \end{bmatrix} \qquad D = 0,$$

Show that the transfer function of such a system is computed,

$$\hat{H}(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \dots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}. \tag{5}$$

2. Let $c(s) = c_{n-1}s^{n-1} + \dots + c_0$ and $d(s) = s^n + \dots + a_0$. If s_0 satisfies $c(s_0) = 0$ and $d(s_0) \neq 0$, show for $u(t) = u_0 e^{s_0 t}, u_0 \in \mathbb{R}$, the zero-state response of the system does not contain a term involving $e^{s_0 t}$.

3. Suppose s_0 is not an eigenvalue of A and that $c(s_0) = 0$. Show that the matrix,

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}, \tag{6}$$

is singular at $s = s_0$. *Hint: determinant.*

2.2 A Skew-Symmetric Stability Condition

Consider the input-free linear, time-varying system $\dot{x}(t) = A(t)x(t)$, where $A(\cdot) \in PC(\mathbb{R}, \mathbb{R}^{n\times n})$. Show that if $A(t)$ is *skew-symmetric* for all $t \in \mathbb{R}$ ($A(t) = -A^\top(t)$), then $x_e = 0$ is a Lyapunov stable equilibrium point.

2.3 Robustness of Exponential Stability

In this problem, we'll show that exponential stability is *robust* under small perturbations.

1. (★ **Hard**—you can skip this subproblem if you can't find a solution after giving it some thought) Consider a family of polynomials parameterized by t,

$$f(s, t) = a_n(t)s^n + \dots + a_1(t)s + a_0(t), \tag{7}$$

where each $a_i : \mathbb{R} \to \mathbb{R}$ is continuous. Prove there exist continuous functions $\lambda_i : \mathbb{R} \to \mathbb{C}$, $i = 1, \dots, n$, such that for all $t_0 \in \mathbb{R}$, each $\lambda_i(t_0)$ corresponds to a root of $f(s, t_0)$.

2. Prove there exists a continuous function $\text{spec} : \mathbb{R}^{n\times n} \to \mathbb{C}^n$, mapping a matrix $A \in \mathbb{R}^{n\times n}$ to a vector containing its eigenvalues.

3. Let $A \in \mathbb{R}^{n \times n}$. Consider the perturbed systems,

$$\dot{x}(t) = (A + \Delta)x(t), \quad x[k+1] = (A + \Delta)x[k], \tag{8}$$

where $\Delta \in \mathbb{R}^{n \times n}$. Suppose each system is globally exponentially stable for $\Delta = 0$. In each case, prove there exists an $M > 0$ such that for all $\Delta : \|\Delta\| < M$, the system remains globally exponentially stable.

2.4 Constant Norm & Constant Speed Systems

The system $\dot{x} = Ax$ is called *constant norm* if, for every trajectory x, $\|x(t)\|$ is constant. The system is called *constant speed* if for every trajectory x, $\|\dot{x}(t)\|$ is constant.

1. Find the (general) conditions on A under which the system is constant norm.
2. Find the (general) conditions on A under which the system is constant speed.
3. Is every constant norm system a constant speed system? Provide a proof or counterexample.
4. Is every constant speed system a constant norm system? Provide a proof or counterexample.

2.5 Separating Hyperplane for a Linear Dynamical System

Let $c \in \mathbb{R}^n$ be a nonzero vector. The *hyperplane* passing through 0 defined by c is the set,

$$\mathcal{H}_c = \{x \in \mathbb{R}^n : c^\top x = 0\} \subseteq \mathbb{R}^n. \tag{9}$$

Consider a continuous-time, LTI system $\dot{x}(t) = Ax(t)$, where $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$. A hyperplane $\mathcal{H}_c$ passing through zero is said to be a *separating hyperplane* for this system if no trajectory of the system ever crosses the hyperplane. That is, if $c^\top \varphi(t, t_0, x_0) < 0$ for some $t \in \mathbb{R}$, it is impossible to have $c^\top \varphi(t', t_0, x_0) > 0$ for another time $t' \in \mathbb{R}$. Assuming the eigenvalues of A are all distinct, explain how to find *all* separating hyperplanes of $\dot{x}(t) = Ax(t)$. Find the conditions on A under which there are *no* separating hyperplanes.

3 Optional Problems

3.1 Stability & System Relations

In this problem, we'll examine how the stability of systems is preserved under coordinate transforms. Here, we'll consider an arbitrary (continuous or discrete-time) input-free system with state transition map $\varphi : \mathbf{T} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\mathbf{T} = \{(t_1, t_0) \in \mathcal{T} \times \mathcal{T} : t_1 \geq t_0\}$.

1. Let $\hat{\varphi} : \mathbf{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be the state transition map of a second, input-free system on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The systems φ and $\hat{\varphi}$ are said to be T-related if,

$$T(\varphi(t, t_0, x_0)) = \hat{\varphi}(t, t_0, T(x_0)), \ \forall t \geq t_0 \in \mathcal{T}, \ x_0 \in \mathbb{R}^m. \tag{10}$$

   If the systems have the form $\dot{x} = A(t)x$ and $\dot{\hat{x}} = \hat{A}(t)\hat{x}$, find sufficient conditions on A and $\hat{A}$ such that the two systems are T-related.

2. Prove that if T is invertible, then the equilibrium $x_e = 0$ of $\dot{x}(t) = A(t)x(t)$ is (Lyapunov/asymptotically/exponentially) stable if and only if the equilibrium $\hat{x}_e = 0$ of $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t)$ is (Lyapunov/asymptotically/exponentially) stable.
3. Suppose now that T is a surjective linear mapping from $\mathbb{R}^n \to \mathbb{R}^k$. What can you conclude about the stability of $\hat{x}_e = 0$ from the stability of $x_e = 0$? What about the case where T is injective? Back up your claims with proofs or counterexamples.