CDS 131 Homework 5: Controllability & Observability

Winter 2025

Due 2/10 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Basic Controllability & Observability Properties

Consider a continuous-time, LTI system representation $(A, B, C, 0)$.

1. Show that if (A, C) is an observable pair, then $C\exp(At)x = 0$ for all $t \in \mathbb{R}$ if and only if $x = 0$. Is the converse of this result true? Provide a proof or a counterexample.
2. Can you restate a version of part (1) for the case when (A, B) is a controllable pair? If so, provide a statement and a proof. If not, explain why not.

1.2 Piecewise Constant Control

The *double integrator* is the continuous-time system with state equation,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{1}$$

Find a *piecewise constant* control strategy which drives the system from the origin to the state $(1, 1)$.

1.3 Transformation into Controllability Form

Using a change of basis, transform the pair,

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2}$$

into the form $\hat{A} = \begin{bmatrix} \hat{A}_c & \hat{A}_{12} \\ 0 & \hat{A}_{uc} \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} \hat{B}_c \\ 0 \end{bmatrix}$, where $(\hat{A}_c, \hat{B}_c)$ is controllable. Is (A, B) a stabilizable pair?

1.4 A Simple Pole-Zero Cancellation

Consider the continuous-time, LTI, SISO system

$$\dot{x}(t) = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \\ c & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ d \\ 1 \end{bmatrix} u(t), \; y(t) = \begin{bmatrix} e & 0 & 0 \end{bmatrix} x, \; e > 0. \tag{3}$$

Compute the transfer function of the system. Using the controllability and observability matrices, show that a pole-zero cancellation in the transfer function takes place if the system loses controllability or observability.

2 Required Problems

2.1 A Diagonal Controllability Condition

Consider the linear, time-invariant system with state equation,

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} u(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$. Show that the system is controllable if and only if $\lambda_i \neq \lambda_j, \; i \neq j$, and $b_i \neq 0$.

2.2 Minimum Energy Required to Leave Safe Operating Region

Consider the linear, time-invariant system $\dot{x} = Ax + Bu, x(0) = 0$, where $x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^m$. Suppose A is Hurwitz and that (A, B) is a controllable pair. Consider the following scenario. Suppose we *don't* have any control over the value of the input signal $u(\cdot)$, but we have some idea of how large its total energy,

$$\|u\|_2^2 = \int_0^\infty \|u(\tau)\|_2^2 \, d\tau, \tag{5}$$

is likely to be. The *safe operating region* for the system is the ball,

$$\mathcal{B} = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}. \tag{6}$$

The hope is that the (unknown) input signal $u(\cdot)$ will not drive the state of the system outside the safe operating region. One measure of system security that is used is the minimum energy $E_{\min}$ that is required to drive the state outside the safe operating region. Fixing $x_0 = 0$ for simplicity, we define,

$$E_{\min} = \inf_{t \in \mathbb{R}_{>0}, u(\cdot) \in \mathcal{U}} \int_0^t \|u(\tau)\|_2^2 \, d\tau \text{ s.t. } \varphi(t, 0, 0, u(\cdot)) \notin \mathcal{B}. \tag{7}$$

Notice that we *do not* fix the time t at which the state leaves the safe operating region—rather, t is an optimization variable. If $E_{\min}$ is much larger than the energy of the unknown input signals we can expect, we can be fairly confident that the state will not leave the safe operating region.

1. Fix an $x \in \mathbb{R}^n$. Recall that, for each $t \in \mathbb{R}_{>0}$, the solution of the optimization problem,

$$\arg\min_{u(\cdot) \in \mathcal{U}} \int_0^t \|u(\tau)\|_2^2 \, d\tau, \text{ s.t. } x(0) = 0, x(t) = x, \tag{8}$$

is given by the input signal $u(\tau) = B^\top e^{A^\top (t-\tau)} W_c(t)^{-1} x$, where $W_c(t)$ is the controllability Gramian at time t. Using this fact, show that the infimum,

$$\inf_{t \in \mathbb{R}_{>0}, u(\cdot) \in \mathcal{U}} \int_0^t \|u(\tau)\|_2^2 \, d\tau, \text{ s.t. } x(0) = 0, \; \varphi(t, 0, 0, u(\cdot)) = x, \tag{9}$$

is equal to $x^\top W_c^{-1} x$, where $W_c = \lim_{t \to \infty} W_c(t)$ is the infinite-horizon controllability Gramian.

2. Using your answer to part (1), calculate $E_{\min}$. Your solution should be in terms of the matrices A, B, or other matrices derived from them such as the controllability matrix $\mathcal{C}$, the infinite-horizon controllability Gramian W_c, and its inverse $P = W_c^{-1}$. Simplify your answer as completely as you can.

3. Suppose the safe operating region is the unit cube $C = \{x \in \mathbb{R}^n : |x_i| \leq 1,\ i = 1, ..., n\}$ instead of the unit ball $\mathcal{B}$. Let $E_{\min}^{\text{cube}}$ denote the minimum energy required to drive the state outside the unit cube C. Repeat part (2) for $E_{\min}^{\text{cube}}$. Once again, simplify your answer as completely as you can.

2.3 Balanced Realizations

In this problem, we'll consider a transformation of a system which enjoys special properties with respect to the controllability and observability Gramians.

1. First, we'll prove a linear algebraic fact. Show that, for positive definite matrices $X, Y \in \mathbb{S}^n, X, Y \succ 0$, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ for which

$$TXT^\top = (T^\top)^{-1} Y T^{-1} = \Sigma, \tag{10}$$

where $\Sigma \in \mathbb{S}^n$ is a diagonal, positive definite matrix. *Hint: examine* $X^{1/2} Y X^{1/2}$.

2. Recall that, under a change of basis matrix $T \in \mathbb{R}^{n\times n}$, a system representation $(A, B, C, 0)$ will transform to $(\hat{A}, \hat{B}, \hat{C}, 0) = (T^{-1}AT, T^{-1}B, CT, 0)$. Suppose A is Hurwitz. Calculate the controllability and observability Gramians $\hat{W}_c = \int_0^\infty e^{\hat{A}t}\hat{B}\hat{B}^\top e^{\hat{A}^\top t} dt, \hat{W}_o = \int_0^\infty e^{\hat{A}^\top t}\hat{C}^\top \hat{C} e^{\hat{A}t} dt$ of the transformed system in terms of the controllability and observability Gramians W_c and W_o of the original system.

3. Show that there exists a change of basis matrix $T \in \mathbb{R}^{n\times n}$ for which the controllability and observability Gramians of the transformed system representation $(\hat{A}, \hat{B}, \hat{C}, 0)$ are equal and diagonal,

$$\hat{W}_c = \hat{W}_o = \operatorname{diag}(\sigma_1, ..., \sigma_n),\ \sigma_i > 0. \tag{11}$$

The constants $\sigma_1, ..., \sigma_n$ are called the *Hankel singular values* of the system, and the representation $(\hat{A}, \hat{B}, \hat{C}, 0)$ is said to be a *balanced realization*. This transformation leads to a popular method of model reduction based on eliminating components corresponding to small Hankel singular values.

3 Optional Problems

3.1 Controllability Miscellanea

Consider a linear, time-invariant system with state equation $\dot{x}(t) = Ax(t) + Bu(t), x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$. For each of the following statements, provide either a proof or a counterexample.

1. Suppose the pair (A, B) is controllable. Is the pair (A^2, B) controllable?

2. Suppose the pair (A^2, B) is controllable. Is the pair (A, B) controllable?

3. Suppose (A, B) is controllable. For a nonzero initial condition $x(0) = x_0 \neq 0$, is it possible to find a piecewise continuous input $u(\cdot) \in PC(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ such that the system is *brought to rest* at $t = 1$ (i.e. $x(t) = 0$ for all $t \geq 1$)?

4. Suppose (A, B) is controllable. Fix a state $\overline{x} \in \mathbb{R}^n$. If the system is initially at rest ($x(0) = 0$), does there exist a piecewise continuous input $u(\cdot) \in PC(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ for which $x(t) = \overline{x}$ for all $t \geq 1$?