CDS 131 Homework 6: State & Output Feedback

Winter 2025

Due 2/18 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*
2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 Practice with Eigenvalue Assignment

In this problem, we'll get some practice with designing controllers via eigenvalue assignment (pole placement). Consider the linear, time-invariant system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{1}$$

1. Using a controllability test of your choice, verify that the system is controllable.
2. Find a static full-state feedback law $u = Kx$ such that the eigenvalues of the closed-loop system are at $\lambda_1 = -2$ and $\lambda_2 = -2$.

1.2 Luenberger Observer Design

In this problem, we'll get some practice designing an observer for a linear system. Consider the linear, time-invariant system,

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \tag{2}$$

Design a Luenberger observer for the system by calculating an observer gain matrix L for which $A + LC$ is Hurwitz. You may choose the observer eigenvalues as you see fit.

1.3 A Simple Tracking Controller

Consider the n'th order system,

$$x^{(n)}(t) = u(t), \tag{3}$$

where $x(t), u(t) \in \mathbb{R}$. Suppose we're given an n-times continuously differentiable trajectory, $x_d(\cdot) \in C^n(\mathbb{R}, \mathbb{R})$, and that we'd like $x(t) \to x_d(t)$ as $t \to \infty$ for all $x(0) \in \mathbb{R}^n$. Show there exists a (potentially time-varying) control law which accomplishes this task.

2 Required Problems

2.1 Feedback Equivalence

In this problem, we'll consider a relationship between two systems called *feedback equivalence*. Consider two pairs of matrices (A, B) and $(\tilde{A}, \tilde{B})$, where $A, \tilde{A} \in \mathbb{R}^{n\times n}$ and $B, \tilde{B} \in \mathbb{R}^{n\times m}$. The pairs are said to be *feedback equivalent*, denoted $(A, B) \equiv (\tilde{A}, \tilde{B})$, if there exist invertible matrices $T \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times m}$, and a matrix $K \in \mathbb{R}^{m\times n}$ for which

$$T^{-1}(A + BK)T = \tilde{A} \tag{4}$$

$$T^{-1}BV = \tilde{B}. \tag{5}$$

Let's study some of the basic properties of feedback equivalent pairs.

1. First, we'll show that feedback equivalence is a *symmetric* relationship between pairs of matrices. Show that if $(A, B) \equiv (\tilde{A}, \tilde{B})$, then $(\tilde{A}, \tilde{B}) \equiv (A, B)$.
2. Show that if $(A, B) \equiv (\tilde{A}, \tilde{B})$, then (A, B) is a controllable pair if and only if $(\tilde{A}, \tilde{B})$ is.
3. Suppose we apply a control law $u = Kx + v$ to a linear, time-invariant system with state equation $\dot{x} = Ax + Bu$. Calculate the new state equation with input v. Under what conditions is the new system with input v controllable?
4. Now, we consider the linear, time-invariant system with state equation $\dot{x} = Ax + Bu$ and output equation $y = Cx + Du$. Suppose we apply a control law $u = Kx + v$ to the system. Calculate the new state and output equations with input v. If (A, C) is an observable pair, is it true that the new system with input v is observable? Provide a proof or counterexample to support your claim.

2.2 Lyapunov-Based Stabilization

In this problem, we'll show that, using a special Lyapunov equation, we can stabilize any controllable, MIMO system using static state feedback. We'll consider the LTI system with state equation,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

1. First, we'll prove an intermediate Lyapunov result. Suppose (A, B) is a controllable pair. Show that A is Hurwitz if there exists a matrix $P \in \mathbb{S}^n, P \succ 0$ for which

   $$AP + PA^\top \preceq -BB^\top. \tag{7}$$

   Hints: $BB^\top$ is not necessarily positive definite! Be careful with A and $A^\top$.
2. Now, we return to the system $\dot{x}(t) = Ax(t) + Bu(t)$, where (A, B) is a controllable pair and A is not necessarily Hurwitz. Prove that, for any $T > 0$, the static state feedback law,

   $$u = -B^\top \left[\int_0^T e^{-A\tau} BB^\top e^{-A^\top \tau} d\tau \right]^{-1} x = -B^\top P^{-1} x, \tag{8}$$

renders the closed-loop system globally exponentially stable. Conclude that if (A, B) is controllable, there exists a matrix $K \in \mathbb{R}^{m \times n}$ for which $A + BK$ is Hurwitz. *Hint: compute* $AP + PA^\top$.

3. Using your answer to part (2), show that if (A, B) is a stabilizable pair, there exists a $K \in \mathbb{R}^{m \times n}$ for which $A + BK$ is Hurwitz.

2.3 A Method for Rapidly Driving the State to Zero

Consider the discrete-time, LTI system with state equation,

$$x[k+1] = Ax[k] + Bu[k], \tag{9}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times k}$, $k < n$, is full rank. The goal is to choose an input u that causes $x[k] \to 0$ as $k \to \infty$ from any initial condition $x[0] = x_0 \in \mathbb{R}^n$. Your friend proposes the following simple method: at time k, choose $u[k]$ to minimize the ℓ_2-norm of the next state, $\|x[k+1]\|_2$. They argue that this scheme will work well, since the norm of the next state is made as small as possible at every time step. In this problem, we will analyze this scheme.

1. Find an explicit expression for the proposed input $u[k]$ in terms of $x[k]$, A, and B. Conclude that the optimal control law is a static, full-state feedback control law.

2. Does the proposed controller always achieve the design goal? Provide a proof or a counterexample to support your claim.

2.4 Deadbeat State & Output Feedback

In this problem, we'll consider a special type of controller for a discrete-time system called a *deadbeat controller*. Such a controller drives a system to the origin as quickly as possible. Let's study the design of a deadbeat controller for a MIMO, discrete-time system,

$$x[k+1] = Ax[k] + Bu[k] \tag{10}$$

$$y[k] = Cx[k] + Du[k], \tag{11}$$

in which (A, B) is a controllable pair and (A, C) is an observable pair.

1. Let's start by considering a deadbeat controller with *full access* to the state of the system. Show there exists a static state feedback control law $u[k] = Kx[k]$ which drives the system to the origin from any initial condition in no more than n steps.

2. Now, we'll consider an output feedback deadbeat controller. Using a Luenberger observer, design an output feedback controller which drives the system to the origin in a finite number of steps. Find an upper bound on the number of steps it takes to drive the state to the origin from any initial condition.

3. Comment on the potential advantages and disadvantages of using a deadbeat controller. When might a deadbeat controller be the best choice? When might it be an impractical choice?

3 Optional Problems

3.1 Lyapunov Condition for Passivity

A *nonlinear* system $\dot{x} = f(x, u), y = h(x)$, $x(0) = 0$ with $u(t), y(t) \in \mathbb{R}^m$ is said to be *passive* if,

$$\int_0^t u(\tau)^\top y(\tau) d\tau \geq 0, \tag{12}$$

holds for all trajectories of the system, and for all $t \geq 0$.

1. Establish the following Lyapunov condition for passivity: if there exists a function V such that $V(x) \geq 0$ for all x, $V(0) = 0$, and $\dot{V}(x, u) \leq u^\top h(x)$ for all u and x, then the system is passive.

2. Now, suppose the system is $\dot{x} = Ax + Bu, y = Cx$, and consider the quadratic Lyapunov function $V(x) = x^\top Px$. Express the conditions in part (1) as a set of matrix inequalities involving A, B, C, P.

3. Consider the system $\dot{x} = Ax + Bu + B_w w$, $y = Cx$. Propose a method of designing a static state feedback controller $u = Kx$ for which the closed-loop system is passive from w to y. Discuss the computational tractability of your proposed method, and the assumptions required for it to work.