CDS 131 Homework 8: Feedback Systems & Internal Stability

Winter 2025

Due 3/5 at 11:59 PM

Instructions

This homework is divided into three parts:

1. *Optional Exercises: the exercises are entirely optional but are recommended to be completed before looking at the problems. They consist of easier, more computational questions to help you get a feel for the material.*

2. *Required Problems: the problems are the required component of the homework, and might require more work than the exercises to complete.*

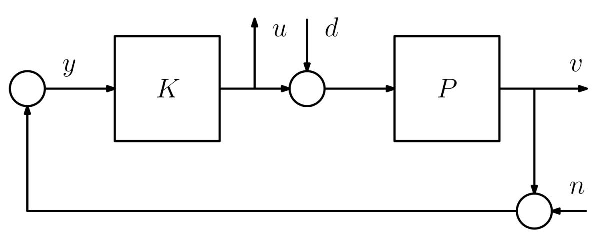
3. *Optional Problems: the optional problems are some additional, recommended problems - some of these might go a little beyond the standard course material.*

All you need to turn in is the solutions to the required problems - the others are recommended but not required.

1 Optional Exercises

1.1 A Simple Feedback Loop

In this problem, we'll show that a simple feedback control system can be recast in the *general feedback arrangement*. Consider the simple feedback system,



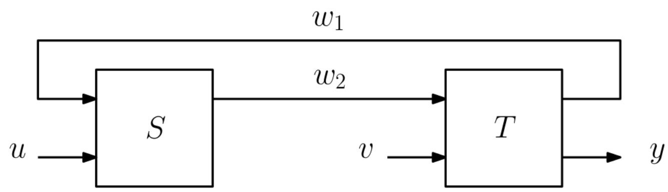
Here, the blocks K and P are continuous-time, LTI systems representing the controller and plant, respectively. The signals $u(t) \in \mathbb{R}^m$ and $v(t) \in \mathbb{R}^p$ together form the controlled output, while the signals $d(t) \in \mathbb{R}^m$ and $n(t) \in \mathbb{R}^p$, which represent a plant disturbance and a measurement noise, respectively, form the exogenous input w. Each circle represents a *summing junction*, in which signals entering the junction are added to form the signal leaving the junction.

Let's rewrite the system in a general feedback arrangement. Suppose P has a state space representation (A, B, C, D) and K has a state space representation (A_K, B_K, C_K, D_K). For $z = (v, u)$ and $w = (d, n)$, identify the linear, time-invariant system representation $G = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ for which the feedback system above is equivalent to the general feedback arrangement interconnecting G and K.

2 Required Problems

2.1 Well-Posedness of a Feedback Interconnection

In this problem, we'll consider a feedback interconnection:



in which S and T are continuous-time, LTI systems with state and output equations,

$$S : \begin{cases} \dot{x} = Ax + B_1 u + B_2 w_1 \\ w_2 = Cx + D_1 u + D_2 w_1, \end{cases} \qquad T : \begin{cases} \dot{z} = Fz + G_1 v + G_2 w_2 \\ w_1 = H_1 z \\ y = H_2 z + J w_2. \end{cases} \tag{1}$$

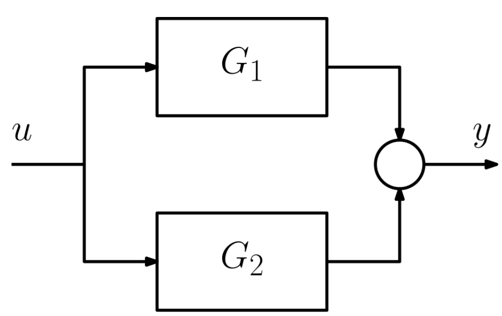
1. Assuming the dimensions of each signal are compatible, express the feedback interconnection as a single continuous-time, LTI system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with input $\tilde{u} = (u, v)$, state $\tilde{x} = (x, z)$, and output y. What additional assumptions (if any) are required for the feedback interconnection to be well-posed?

2. If the systems S and T were now discrete-time, LTI systems, would anything change about the system representation $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of the feedback interconnection? Explain why or why not.

2.2 Stabilizability & Detectability Under Interconnection

In this problem, we'll consider how stabilizability and detectability are affected by system interconnections. Recall that in a block diagram, a circle represents a *summing junction*; a summing junction takes in a number of signals and outputs their *sum*. It is implicit that all signals entering a summing junction are of the same dimension.

1. Consider the *parallel interconnection*:

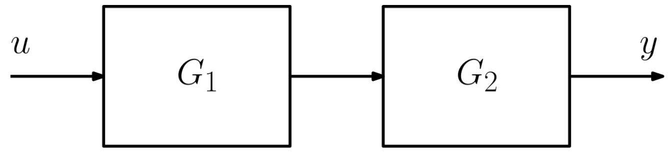


   where the systems $G_1 = (A_1, B_1, C_1, D_1)$ and $G_2 = (A_2, B_2, C_2, D_2)$ are continuous-time, LTI systems. Assuming each signal is of a compatible dimension, derive the state and output equations of the parallel interconnection using a state vector $x = (x_1, x_2)$ (where x_1 is the state of G_1 and x_2 the state of G_2), input vector u, and output vector y.

2. Show that the transfer function from u to y in the parallel interconnection is $\hat{G}_1 + \hat{G}_2$, where $\hat{G}_1$ is the transfer function of G_1 and $\hat{G}_2$ is the transfer function of G_2.

3. If G_1, G_2 are stabilizable and detectable, is the parallel interconnection system from part (1) is stabilizable and detectable? Provide a proof or a counterexample to support your claim. *Hint: cancellation.*

4. Consider the *cascade interconnection*:



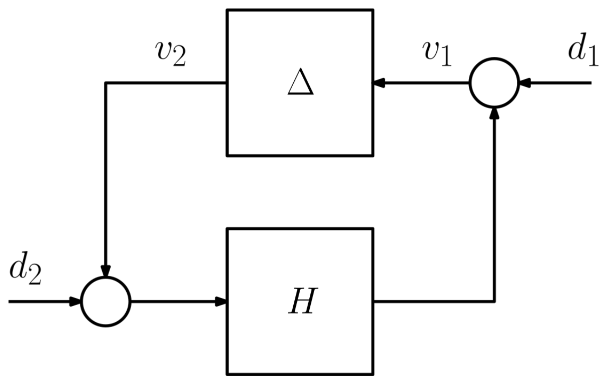
where $G_1 = (A_1, B_1, C_1, D_1)$ and $G_2 = (A_2, B_2, C_2, D_2)$ are continuous-time, LTI systems. Assuming each signal is of compatible dimension, derive the state and output equations of the cascade interconnection using a state vector $x = (x_1, x_2)$, input vector u, and output vector y.

5. Show that the transfer function from u to y in the cascade interconnection is $\hat{G}_2 \cdot \hat{G}_1$, where $\hat{G}_1$ is the transfer function of G_1 and $\hat{G}_2$ is the transfer function of $\hat{G}_2$.

6. If G_1, G_2 are stabilizable and detectable, is the cascade interconnection system from part (4) is stabilizable and detectable? Provide a proof or a counterexample to support your claim.

2.3 The Small Gain Theorem

Consider the following feedback interconnection, in which H and Δ are continuous-time, LTI systems whose transfer functions $\hat{H}, \hat{\Delta}$ belong to $\mathcal{RH}_\infty$.



In the following, you may assume that H and Δ have stabilizable and detectable state space realizations, and that the feedback interconnection is well-posed.

1. Show that if $\hat{G}, \hat{H} \in \mathcal{RH}_\infty$ are transfer functions of compatible dimensions, then $\hat{G}\hat{H} \in \mathcal{RH}_\infty$. *Hint: remember that these transfer functions aren't necessarily SISO!*

2. Show that the transfer function from $(d_1, d_2) \mapsto (v_1, v_2)$ belongs to $\mathcal{RH}_\infty$ if and only if $(I - \hat{H}\hat{\Delta})^{-1} \in \mathcal{RH}_\infty$. *Hint: how can you use well-posedness to justify existence of the inverse?*

3. Recall that the feedback interconnection above is internally stable if the transfer function from $(d_1, d_2) \mapsto (v_1, v_2)$ belongs to $\mathcal{RH}_\infty$. Argue that the feedback interconnection above is internally stable if the function $\det(I - \hat{H}(s)\hat{\Delta}(s))$ has no zeros in the closed right-half plane, $\overline{\mathbb{C}}_+ = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$.

4. Show that $||\hat{\Delta}||_\infty < 1/\gamma$ and $||\hat{H}||_\infty \leq \gamma$ implies internal stability, where $\|\cdot\|_\infty$ is the $\mathcal{H}_\infty$-norm. Give an intuitive explanation as to why this result is true. *Hints: you may as a fact that the $\mathcal{H}_\infty$-norm of a transfer function $\hat{G} \in \mathcal{RH}_\infty$ is equivalently calculated $||\hat{H}||_\infty = \sup_{s \in \overline{\mathbb{C}}_+} \sigma_{\max}(\hat{G}(s))$.*

The result proven in part (4) is one direction of the famous *small gain theorem*, which states that the feedback interconnection above is internally stable for all $\hat{\Delta} \in \mathcal{RH}_\infty$ satisfying $||\hat{\Delta}|| < 1/\gamma$ if and only if $||\hat{H}||_\infty \leq \gamma$.

3 Optional Problems

3.1 The Structured Singular Value

★ This problem is hard, and requires some sophisticated arguments. Give it a try!
Above, we derived a piece of the *small gain theorem*, which is useful in deriving tests for robust stability using the $\mathcal{H}_\infty$-norm of a system Δ representing uncertainty. In the case where the uncertainty has a certain *structure*, however, we can come up with less conservative tests for robust stability! The (complex) structured singular value is a function from the set of $n \times n$ complex matrices to the reals that helps us understand stability in the case of structured uncertainty. In this problem, we will study its basic properties.

The first step towards defining the structured singular value is to define a set of matrices $\vec{\Delta} \subseteq \mathbb{C}^{n\times n}$. Let $r_1, ..., r_S$ and $m_1, ..., m_F$ be positive integers for which $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$. Define a set $\vec{\Delta} \subseteq \mathbb{C}^{n\times n}$ as

$$\vec{\Delta} := \{\text{blkdiag}(\delta_1 I_{r_1}, ..., \delta_S I_{r_S}, \Delta_{S+1}, ..., \Delta_{S+F}) : \delta_i \in \mathbb{C},\ \Delta_{s+j} \in \mathbb{C}^{m_j \times m_j}\}, \tag{2}$$

where I_k represents the $k \times k$ identity matrix. In short, $\vec{\Delta}$ is the set of block diagonal matrices with *repeated scalar blocks* of dimensions $r_i \times r_i$ (these are the blocks $\delta_i I_{r_i}$) and and *full blocks* of dimensions $m_j \times m_j$ (these are the blocks Δ_{S+j}). Given a matrix $M \in \mathbb{C}^{n\times n}$ and a set $\vec{\Delta} \subseteq \mathbb{C}^{n\times n}$ of the form above, one defines the structured singular value of M, $\mu_{\vec{\Delta}}(M)$, as follows.

Definition 1 (Structured Singular Value)**.** *For $M \in \mathbb{C}^{n\times n}$, the structured singular value $\mu_{\vec{\Delta}}(M)$ is defined,*

$$\mu_{\vec{\Delta}}(M) := \frac{1}{\inf\{\sigma_{\max}(\Delta) : \Delta \in \vec{\Delta} \text{ and } \det(I - M\Delta) = 0\}}, \tag{3}$$

unless no $\Delta \in \vec{\Delta}$ makes $I - M\Delta$ singular, in which case $\mu_{\vec{\Delta}}(M) := 0$.

Note that here, we use $\sigma_{\max}(M)$ to denote the maximum singular value of M. Based on this definition, $\mu_{\vec{\Delta}}(M)$ depends both on M and on the set $\vec{\Delta}$. In the following problems, you can assume for simplicity that one does not encounter the case where no Δ makes $I - M\Delta$ singular.

1. Compute $\mu_{\vec{\Delta}}(M)$ in the case where $\vec{\Delta}$ is *unstructured*, i.e. $\vec{\Delta} = \mathbb{C}^{n\times n}$. Then, argue that the structure singular value of a matrix lower bounds its maximum singular value.

2. Recall that the spectral radius of a matrix $M \in \mathbb{C}^{n\times n}$ is defined $\rho(M) := \max_i |\lambda_i(M)|$. Define the set $B_{\vec{\Delta}} = \{\Delta \in \vec{\Delta} : \sigma_{\max}(\Delta) \le 1\}$. Prove that the structured singular value can be calculated,

$$\mu_{\vec{\Delta}}(M) = \sup_{\Delta \in B_{\vec{\Delta}}} \rho(\Delta M). \tag{4}$$

   In the special case where $\vec{\Delta} = \{\delta I_n : \delta \in \mathbb{C}\}$, show that $\mu_{\vec{\Delta}}(M) = \rho(M)$.

3. Let's consider some additional methods of computing μ. Define the following subset of $\mathbb{C}^{n\times n}$:

$$\vec{D} = \{\text{blkdiag}(D_1, ..., D_S, d_{S+1} I_{m_1}, ..., d_{S+F} I_{m_F} : D_i \in \mathbb{C}^{r_i \times r_i}, D_i \succ 0, d_{S+j} \in \mathbb{R}_{>0}\}. \tag{5}$$

   Prove that, for all $D \in \vec{D}$,

$$\mu_{\vec{\Delta}}(M) = \mu_{\vec{\Delta}}(D^{\frac{1}{2}} M D^{-\frac{1}{2}}). \tag{6}$$

   Then, show that,

$$\mu_{\vec{\Delta}}(M) \le \inf_{D \in \vec{D}} \sigma_{\max}(D^{\frac{1}{2}} M D^{-\frac{1}{2}}). \tag{7}$$

4. Fix a matrix $M \in \mathbb{C}^{n\times n}$. For the set $\vec{D}$ introduced in part (3), show that the following set is convex for each fixed $\beta \in \mathbb{R}$:

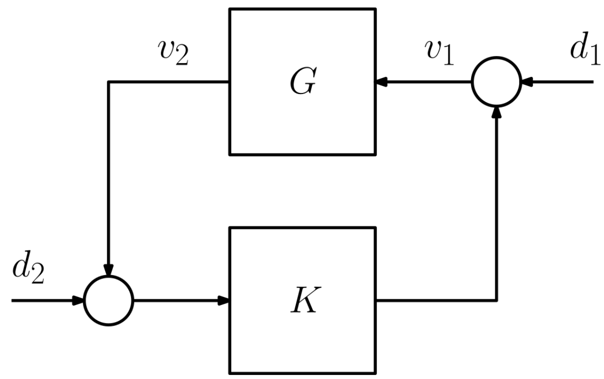
$$\{D \in \vec{D} : \sigma_{\max}(D^{\frac{1}{2}} M D^{-\frac{1}{2}}) \le \beta\}. \tag{8}$$

   Hint: rewrite as a linear matrix inequality. Such inequalities are amenable to implementation in convex optimization solvers!

5. Conjecture a version of the small gain theorem using the structured singular value.

3.2 Controller Parameterization for a Stable Plant

In this problem, we'll study how the stabilizing controller parameterization simplifies when the plant is *already stable*. Consider the following system, in which the transfer function $\hat{G}$ of G belongs to $\mathcal{RH}_\infty$:



You may assume that the interconnection is well-posed (i.e. that $I - DD_K$ is nonsingular, where D is the D matrix of the system representation $G = (A, B, C, D)$ and D_K that of $K = (A_K, B_K, C_K, D_K)$).

1. Show that the feedback system is internally stable if and only if $\hat{K}(I - \hat{G}\hat{K})^{-1} \in \mathcal{RH}_\infty$.

2. Show that for any $\hat{Q} \in \mathcal{RH}_\infty$ for which $(I + \hat{G}\hat{Q})^{-1}$ exists, a controller with a transfer function

$$\hat{K}(s) = \hat{Q}(s)(I + \hat{G}(s)\hat{Q}(s))^{-1}, \tag{9}$$

internally stabilizes the system. *Hint: which transfer function characterizes internal stability?*

3. Prove that if $\hat{K}$ is the transfer function of a stabilizing controller, then there exists a $\hat{Q} \in \mathcal{RH}_\infty$ for which $\hat{K}(s) = \hat{Q}(s)(I + \hat{G}(s)\hat{Q}(s))^{-1}$. *Hint: how can the condition for well-posedness be used?*